DIFFERENTIAL EQUATIONS

I. Introduction

What is a differential equation? An equation for one or more unknown functions involving derivatives of these unknowns, e.g.,

1. $y'' + xy + 3y^2 = e^x$;
2. $u' + u = \cos(t)$;
3. $u_{xx} + u_{yy} = 0$;
4. $x' = y$, $y' = -x$.

If the unknowns are functions of only one independent variable, such as in (1), (2), and (4), we call the equation *ordinary*, otherwise, as in (3), we call the equation a partial differential equation.

Dynamics. $F = ma$. If $x = x(t)$ is the position of the particle at time t, then this equation reads $x'' = F/m$. If also we are given a law for the force $F = F(x,t)$, then this gives a differential equation: $x'' = F(x,t)/m$, e.g., in a constant gravitational field we have $F = -mg$. If $\mathbf{x} = \mathbf{x}(t)$ is a vector in 3-space, and the force is also a vector $\mathbf{F} = \mathbf{F}(\mathbf{x}, t)$, then we have a system of 3 equations in 3 unknowns written as:

$$\mathbf{x}'' = \frac{1}{m}\mathbf{F}(\mathbf{x}, t).$$

In the central force problem we assume that $\mathbf{F} = f(|\mathbf{x}|)\mathbf{x}$. For the central force problem in celestial mechanics, we assume $f(|\mathbf{x}|) = -|\mathbf{x}|^{-3}$, and we get the equations:

$$\mathbf{x}'' = -\frac{\mathbf{x}}{|\mathbf{x}|^3}.$$

This equation governs the motion of the planets around the sun in the first approximation. Suppose that the motion is radial, i.e., along a line from the center, then we get:

$$r'' = -\frac{1}{r^2},$$

where $r = |\mathbf{x}|$.

Heat Diffusion. Let D be a domain in 3-space, say a gas vessel, and let u be temperature in D, then, we have:

$$u_t = u_{xx} + u_{yy} + u_{zz}.$$

If the temperature is in steady state then $u_t = 0$, and we get

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

These are partial differential equations for u.

Calculus of Variations. Many differential equations in physics and engineering are derived from a *variational principle*: the condition that some integral quantity is at a minimum. As an example, we will consider Plateau's problem: finding a surface of least area spanning a given contour. Plateau was a 19th century physicists who experimented with soap films. If one dips a contour made of wire into a soap solution, the surface tension will cause a soap film with least surface area to span the contour when the wire is emerged. Consider the simple special case: when the contour consists of two parallel congruent rings of radius 1 a distance $2l$ apart. Assume that the surface has the same symmetry as the contour, hence is a surface of revolution. The surface can then be described by its generating curve $y = y(x)$. It is convenient to scale the parameters x and y so that the rings are at the location $x = -1$ and $x = 1$. The radius of the rings is then $1/l$.

The surface area of the surface of revolution is given by:

$$A(y) = 2\pi l^2 \int_{-1}^{1} y\sqrt{1+(y')^2}dx.$$

Consider a one-parameter family $y(x)+tz(x)$, where $z(x)$ is a *variation*, an arbitrary continuously differentiable function. If y has least A, then $A(y+tz)$, considered as a function of t, will have a minimum when $t=0$. Differentiate $A(y+tz)$ with respect to t, and set $t=0$ to obtain the directional derivative of A at y in the direction of the variation z:

$$\frac{d}{dt}\left(A(y+tz)\right)|_{t=0} = \int_{-1}^{1}(zF_1+z'F_2)dx,$$

where F_1 and F_2 are expression involving only y and y'. Integrate the second term by parts. There are no boundary terms since z vanishes at the end points; indeed, $z(1)=z(-1)=0$ if all the members of the family $y+tz$ are to satisfy the boundary conditions at $x=1$ and $x=-1$. We obtain:

$$\frac{d}{dt}\left(A(y+tz)\right)|_{t=0} = \int_{-1}^{1} z(F_1+F_2')dx = \int_{-1}^{1} zF\,dx,$$

where F involves only y, y', and y''. Since this is to vanish for all variations z, it follows that $F=0$. This gives the following differential equation for y:

$$yy''-(y')^2-1=0.$$

This differential equation is a necessary condition for y to have least A.

The *order* of an equation is the highest derivative which appears in the equation, e.g.,

1. $y''+y=0$ is a second order equation;
2. $y'=y^2$ is a first order equation.

A *solution* of a differential equation is any function which satisfies the equation, e.g.,

1. $y=\cos(x)$, is a solution of $y''+y=0$;
2. For any constant c, $y=2+ce^{-x}$ is a solution of $y'+y=2$;
3. For any constant a, $y=\cosh(ax)/a$ is a solution $yy''-(y')^2-1=0$.

The general solution of a differential equation is a solution written in terms of a number of arbitrary constants such that every solution of the equation can be obtained by setting the constant to some value,

1. $y=\cos(x)$ is <u>not</u> the general solution of $y''+y=0$; $y=\sin(x)$ is another solution;
2. $y=2+ce^{-x}$ is the general solution of $y'+y=2$;
3. $y=\cosh(ax)/a$ is not the general solution of $yy''-(y')^2-1=0$; $y=\cosh(ax+b)/a$ is the general solution;

Sometimes a solution cannot be written explicitly, but may be given implicitly as the solution of an (algebraic) equation, e.g.,

1. $y^2+x^2=1$ defines implicitly a (actually two) solutions of $yy'=-x$;
2. $e^y+xy=c$ defines implicitly a solution of $(e^y+x)y'+y=0$.

The graph of a solution is called an *integral curve* of the equation. The set of integral curves is often more important than any single solution. For "regular" equations of the form $y'=f(x,y)$, where f is "nice", integral curves cannot cross, e.g.,

1. The set of curves $y + ce^{-x}$ for various values of c, are the integral curves of the equation $y' + y = 2$;
2. The set of curves
$$y = e^{x^2/2}\left(\int_0^x 2e^{s^2/2}\,ds + c\right),$$
are the integral curves of $y' + xy = 2$.

The general solution of an equation of first order always contains one arbitrary constant. This is often determined by *initial conditions*, e.g.,

1. The problem $y' + y = 2$, $y(0) = 4$, has the unique solution $y = 2 + 2e^{-x}$;
2. The problem $y' + xy = 2$ has the unique solution:
$$y = e^{x^2/2}\left(\int_0^x 2e^{s^2/2}\,ds + 1\right).$$

The general solution of a second order equation contains two arbitrary constants. To determine the solution uniquely, two initial conditions must be given, e.g., $r'' = -r^{-2}$, $r(0) = r_0$, $r'(0) = v_0$. Sometimes boundary conditions are given instead of initial conditions, e.g., $yy'' - (y')^2 - 1 = 0$, $y(1) = y(-1)1/l$.

A concept which is sometimes useful to obtain a rough idea about the geometry of the integral curves is the *direction field.* For an equation in the form $y' = f(x, y)$, the right hand side gives the slope of the integral curve at the point (x, y) in the plane. The line through (x, y) with slope $f(x, y)$ is called a *lineal element* for the equation. By drawing a dense collection of those lineal elements, one may obtain some qualitative notion on the shape of the integral curves. *Caution:* Although the direction fields of the equations $y' = y$, and $y' = y^2$ are relatively similar, their integral curves differ a great deal!

Homework: 1, 3, 7, 9, 12, 14, 15, 18, 20, 24, 25, 28, 30, 33, on pp. 9-10.

II. Separable Equations

An ordinary differential equation is separable if it can be written as $y' = A(x)\,B(y)$. We separate the variables:

$$\frac{dy}{B(y)} = A(x)\,dx,$$

and integrate to get an implicit solution.

1. $y' = y^2e^{-x}$. Assuming $y \neq 0$, we have $y^{-2}\,dy = e^{-x}dx$, which after integration leads to $y = 1/(e^{-x} - c)$. Since $y = 0$ is a solution which is not included (cannot be obtain by setting any value for c), this is not the general solution.
2. $x^2y' = 1 + y$. Assuming $x \neq 0$, and $y \neq 0$, we get $y = -1 + ke^{-1/x}$. Note that $y = 0$ is a solution obtain by setting $k = 0$. Draw a few integral curves.

Carbon Dating. Radioactive decay is governed by the law: $m' = km$. This is a separable equation with the general solution: $y = Ae^{kt}$. Note that $A = y(0)$ is the initial mass. Note also that it takes a fixed time for any amount of radioactive material to decay to half its size. This time period is called the *half-life* of the radioactive material, e.g.,, the C-14 isotope of carbon has a half- life of 5570 years. We have $M/2 = y(H) = Me^{kH}$, hence $k = -H^{-1}\ln 2$, i.e., $y = M\exp(-tH^{-1}\ln 2)$. C-14 is produced by γ-rays in the upper parts of the atmosphere, and absorbed into living organisms. During the organism's life, the concentration of C-14 is in equilibrium. Once an organism dies, it does not absorb C-14 from the atmosphere anymore, and the isotope starts to decay. The amount of C-14 in a sample can be measured by counting β-emissions, e.g., C-14 has an emission of 15 per gram per minute. Suppose, an ancient dead wood sample is emitting 5 β-particles a minute, while a comparable living wood sample would contain 2 grams of C-14. Then $M = 2$, $H = 5570$, and $m = 1/3$. We have to determine t. We get the equation $1/3 = 2\exp(-t\ln(2)/5570)$ which gives $t = 5570\ln(6)/\ln(2) \approx 14,000$ years.

Drainage. Two laws govern drainage:

1. $dV/dt = -kAv$, where V is the volume of the fluid in the tank, A is the cross-sectional area of the drain-hole, v is the velocity of the fluid leaving the tank through the drain, and $0 < k < 1$ is an efficiency constant.
2. *Toricelli's Law* states that v is the velocity of a freely-falling particle released from a height equal to the height of the fluid above the drain hole: $v = \sqrt{2gh}$, where g is the gravitational acceleration.

Put together, these two laws give:

$$\frac{dV}{dt} = -kA\sqrt{2gh}.$$

To obtain a differential equation one must express V and h in terms of one variable. This step depends on the shape of the tank.

Consider for example, a hemi-spherical tank of radius R. By slicing, we have that $V(h) = \int_0^h \pi r^2\,dh = \pi\int_0^h(2Rt - t^2)\,dt$, hence $dV/dt = \pi(2Rh - h^2)dh/dt$. Thus, we obtain the differential equation:

$$\pi(2Rh - h^2)h' = -kA\sqrt{2gh},$$

which is separable, and integrates to:

$$\frac{4}{3}Rh^{3/2} - \frac{2}{5}h^{5/2} = -\frac{kAt}{\pi}\sqrt{2g} + C,$$

where C is a constant of integration. If $R = 18$ feet, the radius of the hole is 3 inches, and $k = 0.8$, we obtain: $60h^{3/2} - h^{5/2} = -t + k$, and k is determined from the condition $h(0) = 18$. We find $k = 2268\sqrt{2}$. The tank is empty when $h = 0$, i.e., when $t = k = 2268\sqrt{2}$.

Escape Velocity. Suppose a projectile is fired from the surface of a 'star' of radius R, with initial velocity v_0. Determine the *escape velocity*, the least value of v_0 necessary for the projectile to escape the gravitational field of the star.

We use the equation for radial motion in a gravitational field: $r'' = -GM/R^2$, where G is the gravitational constant, and M is the mass of the star. The important observation here is the independent variable t does not appear explicitly in the equation. As a consequence, we can transform the equation into a separable first order equation for $v = dr/dt$ as a function of r. Indeed, $dv/dt = (dv/dr)(dr/dt) = v\, dv/dt$. Thus, we obtain: $v\, dv/dt = -GM/r^2$ which integrates to:

$$\frac{1}{2}v^2 - \frac{GM}{r} = E.$$

This identity is called *conservation of energy* in physics. Now, note that the projectile escapes iff $E \geqslant 0$. (Draw the potential energy curve $P = -GM/r$). Thus, the condition for escape is:

$$v_0^2 \geqslant \frac{2GM}{R}.$$

Assuming that light was made of particles whose initial velocity was c, Laplace, in 1796, used this argument to show that if a star is so dense that $GM/Rc^2 > 1$, then no light can escape from that star. Although, the argument is wrong (why?), the condition $2GM/Rc^2 > 1$ is the correct one for the formation of a *black hole* in general relativity!

To integrate the equation further, solve the conservation of energy for $v = dr/dt$, and separate variables again, and integrate:

$$t - c = \pm \int \frac{dr}{\sqrt{2E + 2GM/r}}.$$

Homework: 3, 9, 10, 13, 17, 18, 19, 22, 23, 24, 25, 26, 27, 28, 30, 31, 34, on pp. 18–22.

III. Linear Equations, Exact Equations and Integrating Factors

Linear Equations. A *first order linear differential equation* is one that has the form: $y' + p(x)y = q(x)$. We treat the linear equation by the method of variation of parameters. In this method, we first consider the homogeneous case, in which $q(x) = 0$. In this case, the equation becomes $y' = -p(x)y$ which is separable. The solution is $y = k\exp(-\int p\,dx)$.

1. $y' + y = 0$; $y = ke^{-x}$;
2. $y' + xy = 0$; $y = ke^{-x^2/2}$.

For the non-homogeneous case, replace the constant k by a function, and substitute back into the equation to get k', i.e., try $y = k(x)\exp(-\int p\,dx)$. One gets $k' = q\exp(\int p\,dx)$, integrates $k = \int q\exp(\int p\,dx)\,dx + c$, and substitutes back into y.

1. $y' + y = 2$; trying $y = k(x)e^{-x}$, we get $k' = 2e^x$, $k = 2e^x + c$, and thus $y = 2 + ce^{-x}$;
2. $y' + xy = 2$; trying $y = k(x)e^{-x^2/2}$, we get $k' = 2e^{x^2/2}$, and thus $y = e^{-x^2/2}(\int_0^x 2e^{s^2/2}ds + c)$.

Homework: 2, 4, 6, 13, 14, 23, 25, 27, on pp. 25–26.

Exact Equation. Consider a differential equation written as: $M(x, y) + N(x, y)y' = 0$. This can always be done, e.g., if $y' = f(x, y)$, we take $M = f$, and $N = -1$. A *potential function* for this differential equation is a function function $\varphi(x, y)$ such that $\varphi_x = M$, and $\varphi_y = N$. If such a function exists, we say that the equation is *exact*. In this case, then for any solution $y = y(x)$, we have that $\varphi(x, y(x))$ is constant. Indeed, differentiating implicity:

$$\frac{d}{dx}\left(\varphi(x, y(x))\right) = \varphi_x + \varphi_y\frac{dy}{dx} = M + Ny' = 0.$$

Thus, $\varphi(x, y(x)) = c$ defines implicitly the general solution of the differential equation. For example $\varphi(x, y) = 2y^2x + e^{xy} + y^2$ is a potential function for the differential equation: $2y^2 + ye^{xy} + (4xy + xe^{xy} + 2y)y' = 0$. The general solution of this differential equation is given by $2y^2x + e^{xy} + y^2 = c$.

Theorem 1. *Suppose that M, N, M_y, and N_x are continuous for all (x, y) in a rectangle R. Then the differential equation$M + Ny' = 0$ is exact if and only if $M_y = N_x$.*

Indeed, if the equation is exact, then $M_y = \varphi_{xy} = \varphi_{yx} = N_x$. We illustrate the converse by example: $3y^4 - 1 + 12xy^3y' = 0$. We have $M_y = 12y^3 = N_x$. Integrate $3y^4 - 1$ with respect to x to get $\varphi = (3y^4 - 1)x + k(y)$. Then differentiate with respect to y and set equal to N: $12y^3x + k' = 12xy^3$. We get $k' = 0$, hence $k = 0$, and a $\varphi = (3y^4 - 1)x$.

Integrating factors. Although $M + Ny' = 0$ is not exact, it is possible that $\mu M + \mu Ny' = 0$ is exact for some $\mu \neq 0$. Such a function is called an *integrating factor*. As before a condition for μ to be an integrating factor is $(\mu M)_y = (\mu N)_x$, i.e.,

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x,$$

This is a linear partial differential equation for μ, which may be not be easier to tackle than the original equation. For example, if it happens that $(M_y - N_x)/N$ is independent of y, we may look for $\mu = \mu(x)$, by solving $\mu_x/\mu = (M_y - N_x)/N$. As an example, we consider the first order linear equation $py - q + y' = 0$. We have $M = py - q$, and $N = 1$, hence $(M_y - N_x)/N = p$ is independent of y. Thus, an integrating factor is $\mu = \exp(\int p\,dx)$. Multiplying the equation by μ we get: $(\mu y)' = \mu q$, which can be integrated: $y = \mu^{-1}(\int \mu q\,dx + c)$. Similarly, if $(N_x - M_y)/M$ is independent of x, then we may look for $\mu = \mu(y)$ by solving $\mu_y/\mu = (N_x - M_y)/M$.

Otherwise, some other guess for μ may have to substituted. For example, consider the equation: $x^2y' + xy + y^{-3/2} = 0$. We have $M_y = x - 3y^{-5/2}/2$, $N_x = 2x$, and clearly neither of the above works. We try an integrating factor of the form $\mu = x^a y^b$. We get:

$$(b - 3/2)xy^{-3/2} = (a - b + 1)x^2y,$$

which holds if and only if $b = 3/2$, $a = 1/2$. After multiplying by $\mu = x^{1/2}y^{3/2}$, we get the equation: $x^{5/2}y^{3/2}y' + x^{3/2}y^{5/2} - (3/2)x^{1/2} = 0$, which integrates to: $x^{5/2}y^{5/2} - (5/2)x^{3/2} = c$.

IV. Special Differential Equations and Applications

Homogeneous Equations. A differential equation is *homogeneous* if it can be written as $y' = f(y/x)$, e.g.,

1. $y' = (x/y)\sin(y/x)$;
2. $y' = y/(x+y)$.

The change of variable $u = y/x$, $y' = u + xu'$ transforms the homogeneous differential equation $y' = f(y/x)$ into $xu' = f(u) - u$ a separable differential equation, e.g.,

$$y' = \frac{x}{y} + \frac{y}{x},$$

leads to $uu' = 1/x$, which integrates to $u = \sqrt{2\ln x + c}$, i.e., $y = \pm x\sqrt{2\ln x + c}$.

A Pursuit Problem. A dog jumps into a canal of width w and swims with speed v directly across in a current of speed s. Pick coordinates in the plane so that the origin is the aim while $(w, 0$ is the starting point. Let $x = x(t)$, $y = y(t)$ be parametric equations for the dog's path. Then we have $x' = -v\cos(\varphi)$, $y' = s - v\sin(\varphi)$, where φ is the polar angle of the dog's position. Thus, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dy} = \frac{s - v\sin(\varphi)}{v\cos(\varphi)} = \tan(\varphi) - \frac{s}{v}\frac{1}{\cos(\varphi)}.$$

Since $\tan(\varphi) = y/x$, and $1/\cos(\varphi) = x^{-1}\sqrt{x^2 + y^2}$, we get the differential equation:

$$\frac{dy}{dx} = \frac{y}{x} - \frac{s}{v}\sqrt{1 + \frac{y^2}{x^2}},$$

which is homogeneous. After integrating we get:

$$u + \sqrt{1 + u^2} = cx^{-s/v}.$$

The constant c is determined by the initial condition $y(w) = 0$: $c = w^{x/v}$. Solve for u, and substitute $y = xu$ to get:

$$y = \frac{w}{2}\left[\left(\frac{x}{w}\right)^{1-s/v} - \left(\frac{x}{w}\right)^{1+s/v}\right].$$

Bernoulli Equation. A *Bernoulli equation* is an equation of the form $y' + p(x)y = r(x)y^{\alpha}$, for some real number α. If $\alpha \neq 1$, the substitution $u = y^{1-\alpha}$ transforms the Bernoulli equation into a linear equation, e.g., $x^3 y' = x^2 y - y^3$ is a Bernoulli equation with $\alpha = 3$. Set $u = y^{-2}$, to get: $u' + 2u/x = 2/x^3$, which integrates to $x^2 u = 2\ln x + c$. Substituting back $u = y{-}2$, we get $y = \pm x/\sqrt{2\ln x + c}$.

Riccati Equation. A *Riccati Equation* is an equation of the form $y' = p(x)y^2 + q(x)y + r(x)$. If $u(x)$ is a particular solution of this equation, the substitution $y = u + 1/z$ will yield a linear for z. Thus, the general solution can be found. $y = x$ is a solution of $y' = xy^2 - x^3 + 1$, hence substitute $y = x + 1/z$ to get $z' + 2x^3 z = -x$. Integrating this linear differential equation we get:

$$z = e^{-2x^3/3}\left(c - \int_0^x se^{2s^3/3}\,ds\right),$$

from which we conclude that the general solution of $y' = xy^2 - x^3 + 1$ is

$$y = x + e^{2x^3/3}\left(c - \int_0^x se^{2s^3/3}\,ds\right)^{-1}.$$

Homework: 1, 3, 5, 9, 11, 13, 15, 17, 20, 21, 25, 26, 27, 28, 29, 30, 32, 33, 37, on pp. 43-47.

Mechanics A 16-foot long chain weighing ρ pounds per foot hangs over a small pulley, 20 feet above the ground. The chain is released with 7 feet on one side and 9 feet on the other. How lone after the chain is released and with what velocity will the chain leave the pulley. Denote by $x(t)$ the displacement from equilibrium. The net force on the chain is $2x\rho$, and its mass is $16\rho/32 = \rho/2$. The equation of motion is: $dv/dt = 4x$. Since $dv/dt = v\,dv/dx$, we get $v^2 = 4x^2 + E$, where E is constant. To determine E, set $x = 1$, and $v = 0$. Conclude $E = -4$. When $x = 8$, we get $v = \sqrt{252}$. To get the time required separate variables:

$$dt = \frac{dx}{2\sqrt{x^2-1}},$$

and integrate:

$$t = \int_1^8 \frac{dx}{2\sqrt{x^2-1}} = \frac{1}{2}\ln(8+\sqrt{63}).$$

Suppose instead that the chain is 40-foot long and begins to unwind when released with already 10 feet played out. Determine the velocity when the chain leaves the support. Denote by $x(t)$ the length of the chain which has left the support. The equation of motion is: $m\,dv/dt + v\,dm/dt = F$. We have $F = x\rho = mg$, hence $m = x\rho/32$, and $dm/dt = \rho v/32$. As above $dv/dt = v/\,dv/dx$, and the equation is:

$$v' + \frac{v}{x} = 32v^{-1},$$

a Bernoulli equation. Substitute $v = w^{1/2}$ to get:

$$w' + \frac{2w}{x} = 64,$$

which integrates to $w = v^2 = 64x/3 + c/x^2$. Set $v = 0$, $x = 10$, to get $c = -64,000/3$, hence: $v^2 = (64/3)(x - 1000/x^2)$. When $x = 40$, we have $v^2 = 840$, i.e., $v = 2\sqrt{210}$.

Friction is usually assumed to be proportional to v^2. Thus, the equation of motion for free fall in a constant gravitational field with friction is: $m\,dv/dt = mg - \alpha v^2$. This is a separable equation:

$$\frac{dv}{g-(\alpha/m)v^2} = dt.$$

Integrate to get:

$$\sqrt{\frac{m}{\alpha g}}\tanh^{-1}\left(\sqrt{\frac{\alpha}{mg}}v\right) = t + c.$$

From the initial conditions $v(0) = 0$, we get $c = 0$. Solve for v to get:

$$v(t) = \sqrt{\frac{mg}{\alpha}}\tanh\left(\sqrt{\frac{\alpha g}{m}}t\right).$$

We see that $v \to \sqrt{mg/\alpha}$, the *terminal velocity*, as $t \to \infty$.

Electrical Circuits. *Kirchoff's current law* states that the sum of the currents entering any juncture in a circuit is equal to the sum of the currents leaving the juncture. *Kirchoff's voltage law* states that the sum of the potential drops in a circuit is zero. Consider for example a circuit in which a battery, a resistor, and an inductor are placed in series. We get: $E - iR - Li' = 0$. This is a linear equation for i whose general solution is: $i = E/R + Ke^{-Rt/L}$. The constant K is determined by the initial conditions.

Consider next a circuit in which a battery, a switch, a resistor, and a capacitor are placed in series. Assume that initially, the switch is open, and the charge on the capacitor is zero. If q is the charge on the capacitor, we get: $iR + q/C = E$, where $i = q'$. Thus, the equation is: $q' + q/(RC) = E/R$, whose general solution is: $q = EC(1 - ke^{-t/RC})$. The constant k is determined from the initial condition $q(0) = 0$. Thus, $k = 1$, and the solution of the initial value problem is: $q = EC(1 - e^{-t/RC})$. Note that the voltage $q/C \to E$ as $t \to \infty$, hence in time a negligible amount of current flows in the circuit, and there is a negligible drop in voltage across the resistor. The current is given by $i = q' = Ee^{-t/RC}/R$.

Geometry Consider the family of curves given by: $y^2 = kx^3$, for various values of k. Find a family of orthogonal trajectories. We first derive a differential equation for y. Differentiating, we get $2yy' = 3kx^2$. We substitute $k = y^2/x^3$ in this equation to get: $2yy' = 3y^2/x$, which can be simplified into: $y' = 3y/2x$. The orthogonal family has $y' = -3y/2x$, which is separable and integrates to: $3y^2 + 2x^2 = k^2$, a family of ellipses.

Homework: 1, 2, 4, 10, 11, 12, 13, 15, 17, 18, 21, 22, 24, 26, 27, 29, 31, 32, 35, 39, 41, 43, on pp. 55–59.

V. Existence and Uniqueness

Consider the initial value problem: $y' = f(x,y)$, $y(x_0) = y_0$. This may not always be solvable explicitly. Here, we discuss the solvability of this problem in principle. Note that there may be no solutions, e.g., $y' = 2\sqrt{y}$, $y(0) = -1$. Furthermore, there may be more than one solutions, e.g., $y' = 2\sqrt{y}$, $y(1) = 0$ has both the solutions $y_1 = 0$, and $y_2 = 0$ for $x < 1$, and $y_2 = (x-1)^2$ for $x \geqslant 1$. The solution is not unique. In both of these case the function $f(x,y) = 2\sqrt{y}$ was the problem. We state the following existence and uniqueness theorem:

Theorem 2. *Let f and f_y be continuous in a rectangle R centered at (x_0, y_0). Then, there exists a positive number h such that the initial value problem:*

$$y' = f(x,y), \qquad y(x_0) = y_0,$$

has a unique solution in the interval $|x - x_0| < h$.

Note that $(0,-1)$ is not in any rectangle where $f(x,y) = 2\sqrt{y}$ is defined, whence the first initial value problem above has no solutions, and $f_y = 1/sqrty$ is not continuous in any rectangle containing the point $(1,0)$, whence the second initial value problem above has more than one solution. We also note that the restriction to a small interval $(x_0 - h, x_0 + h)$ is necessary. Indeed, consider the problem $y' = y^2$, $y(0) = 1/a$, for $a > 0$. All the conditions of the theorem are satisfied, and indeed there is a unique solution for any a given by: $y = 1/(a - x)$. However, this solution exists only on the interval $x < a$. The question of whether h can be made arbitrary large is referred to as the problem of *global existence*. One case in which this can be settled is the linear case:

Theorem 3. *Let p and q be continuous on an open interval I, and let y_0 be any real number. Then the initial value problem:*

$$y' + p(x)y = q(x), \qquad y(0) = y_0,$$

has a unique solution defined in I.

Thus, in particular, if p and q are defined for all real x, also the solution exists for all x.

Another important problem in applications is the problem of continuous dependence on the initial data. We need to know that if the data is varied a small amount, the solution only changes by a correspondingly small amount. We will denote by $y(x;a)$ the solution of the initial value problem $y' = f(x,y)$, $y(x_0) = a$. We have the following theorem:

Theorem 4. *Let f and f_y be continuous on a rectangle R centered at the point (x_0, y_0). Suppose the solution $y(x; y_0)$ exists on an interval $|x - x_0| < h$ for some $h > 0$. Then there is a positive number k such that if $|a - y_0| < k$, then $y(x;a)$ also exists in that interval, and $y(x;a) \to y(x;y_0)$ for all x in that interval.*

We conclude this discussion by remarking that although all the statements here have been formulated for a single equation $y' = f(x,y)$, they apply equally well to systems $\mathbf{y}' = \mathbf{F}(x, \mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$.

Homework: 1, 5, 6, 8, on pp. 62–63.

VI. Second Order Linear Equations

A second order linear equation has the form: $y'' + p(x)y'(x) + q(x)y = f(x)$. For example the charge q in an RLC circuit satisties the equation $q'' + (R/L)q' + (1/LC)q = E/L$. As mentioned earlier, the initial data for second order equations consists of initial position and initial velocity.

Theorem 5. *Let p, q, and f be continuous on an open interval I. Let x_0 be a point in I, and let a, and b be real numbers. Then the initial value problem: $y'' + py' + qy = f$, $y(x_0) = a$, $y'(0) = b$ has a unique solution defined for all x in I.*

The existence of a local solution is obtained here, as for all second order equations, by transforming the problem into a first order system. This is done by introducing the variable $z = y'$. Thus, we can write the problem as a system:

$$y' - z = 0, \qquad z' + pz + qy = f, \qquad y(0) = a, \qquad z(0) = b.$$

The solution is now obtained by using the existence theorem for first order systems mentioned in the last section. For a linear systems, one can also write the solution explicitly, which gives global existence.

The Homogeneous Case. We consider now the homogeneous case, i.e., $f = 0$. Note that in this case if y_1 and y_2 are two solutions, then also any *linear combination* $c_1y_1 + c_2y_2$ is a solution. We say that two solutions y_1, and y_2 are *linearly dependent* on an open interval I if either is a constant multiple of the other,e.g., $5\cos(x)$ and $\cos(x)$ are linearly dependent. Otherwise, we say that the solutions are *linearly independent*. In this case, we say that they form a *fundamental set of solutions*, e.g., $\cos(t)$ and $sin(t)$ form a fundamental set of solutions of the equation $y'' + y = 0$. We define the *Wronskian* by:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

To test for linear independence, we have the following.

Theorem 6. *Suppose that p, and q and continuous on I. Let y_1 and y_2 be solutions of the equation $y'' + py' + qy = f$. Then:*

1. *Either $W(x) = 0$ for all x in I, or $W(x) \neq 0$ for any x in I.*
2. *y_1, and $y_@$ are linearly independent if and only if $W(x) \neq 0$ on I.*

To see this, one first verifies that $W' + pW = 0$. Thus $W = \exp(-p\,dx)$ and (1) follows. To see (2), note that we may assume $y_2(x_0) \neq 0$ for some x_0 in I, hence we may assume that $y_2(x) \neq 0$ on some interval J. Since $(y_1/y_2)' = W/y_2^2$, we see that $y_1/y_2 = c$ in J. But then y_1 and cy_2 both satisfy the same initial value problem, and hence are equal. We can now write the *general solution* of the equation as $c_1y_1(x) + c_2y_2(x)$ where c_1 and c_2 are arbitrary constants.

Theorem 7. *Let y_1 and y_2 form a fundamental set of solutions of the equation $y'' + py + q = 0$ on I. Then every solution on I is a linear combination of y_1 and y_2.*

To see this, let φ be a solution, fix x_0 in I, set $a = \varphi(x_0)$, $b = \varphi'(x_0)$, and consider the initial value problem:

$$y'' + py' + qy = f, \qquad y(0) = a, \qquad y'(0) = b.$$

Clearly, φ is a solution. We will show that there is a solution of this problem which is a linear combination of y_1 and y_2. By the uniqueness theorem, this solution must equal φ. Write the initial conditions for $c_1y_1 + c_2y_2$ as:

$$y_1(x_0)c_1 + y_2(x_0)c_2 = a$$
$$y_1'(x_0)c_1 + y_2'(x_0)c_2 = b.$$

This system of two equations in two unknowns has a solution since $W(x_0) \neq 0$.

The Non-Homogeneous Case. The general solution of the non-homogeneous equation is the sum of the general solution of the homogeneous equation with any one particular solution of the non-homogeneous equation. For example, a particular solution of $y'' + y = 2$ is $y = 2$. Thus, the general solution of the non-homogenous equation is $y = c_1\cos(t) + c_2\sin(t) + 2$. This follows from the following.

Theorem 8. *Let φ, and ψ be solutions of the equation $y'' + py' + q' = f$, then $\varphi - \psi$ satisfy the homogeneous equation $y'' + py' + qy = 0$.*

Homework: 3, 5, 7, 8, 11, 12, 13, 15, 16, 18, on pp. 73–74.

Reduction of Order. If one knows a solution φ of the homogeneous equation $y'' + py' + qy = 0$, then one may find a linearly independent solution by subtituting $y = u\varphi$ in the equation. This gives a first order linear equation for u'. This can be shown in general, but we will first illustrate this principle by way of example. Consider the equation $x^2y'' - 7xy + 16y = 0$. The function $y = x^4$ satisfies the equation. We substitute $y = ux^4$ in the equation to get $xu'' + u' = 0$. This can be integrated to $u' = 1/x$, hence $u = \ln(x)$. Thus, a fundamental set of solutions of the equation is x^4 and $x^4\ln(x)$, and the general solutions is $y = c_1x^2\ln(x) + c_2x^4$.

For the general case, suppose that u satisfies the equation $y'' + py' + qy = 0$. Substitute $y = uv$ in this equation, and use the fact that $u'' + pu' + qu = 0$, to get $2u'v' + uv'' + puv' = 0$. Substitute $w = v'$ to get a first order homogeneous linear equation $uw' + (pu + 2u')w = 0$ for w. If c_1w is the general solution of this equation with c_1 an arbitrary constant, then the general solution of the original equation $y'' + py' + qy = 0$ is $y = c_1uv + c_2u$, where $v = \int w\,dx$ is the indefinite integral of w. Indeed, $y_1 = uv$, and $y_2 = u$ form a fundamental set of solutions. The Wronskian is $-u^2v'$.

Homework: 5, 8, 12, 14, 15, 16, 17, 18, 19, on pp. 76–77.

VII. Constant Coefficients Linear Equations

Homogeneous Equations. A linear second order homogeneous equation with constant coefficients has the form $y'' - 2by' + cy = 0$, where b and c are constants. We substitute $y = e^{\lambda x}$ to get $e^{\lambda x}(\lambda^2 - 2b\lambda + c) = 0$. Since $e^{\lambda x}$ is never zero, we must have $\lambda^2 - 2b\lambda + c = 0$. This equation is called the *characteristic equation.* We distinguish three cases:

1. $b^2 > c$; the characteristic equation has two distinct real roots: $\lambda_{1,2} = b \pm \sqrt{b^2 - c}$. In this case the two solutions $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are linearly independent; $W(x) = (\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)x}$. For example, the equation $y'' - y' - 2 = 0$ has characteristic equation $\lambda^2 - \lambda - 2 = 0$ with real roots -1 and 2, hence the general solution is $y = c_1e^{-x} + c_2e^{2x}$.
2. $b^2 = c$; the characteristic equation has exactly one root $\lambda = b$. In this case, $y_1 = e^{bx}$ is one solution, and using reduction of order, another solution is found $y_2 = xe^{bx}$. These two solutions are linearly independent; $W(x) = e^{2bx}$. For example, the general solution of $y'' + 2y' + 1 = 0$ is $y = c_1e^{-x} + c_2xe^{-x}$.
3. $b^2 < c$; the charateristic equation has complex roots $b \pm iq$, where $q = \sqrt{c - b^2}$. Using *Euler's formula* $e^{it} = \cos(t) + i\sin(t)$, and taking the sum and the difference of $e^{(b+iq)x}$ and $e^{(b-iq)x}$, we obtain two linearly independent real solutions $y_1 = e^{bx}\cos(qx)$ and $y_2 = e^{bx}\sin(px)$; $W(x) = qe^{2bx}$. For example, the equation $y'' + y$ has characteristic equation $\lambda^2 + 1 = 0$, with roots $\pm i$. The general solution is $y = c_1\cos(x) + c_2\sin(x)$.

Note that Euler's formula can be justified using power series: substitute $x = it$ in the Taylor expansion of $e^x = \sum_{n=0}^{\infty} x^n/n!$, use $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and separate real and imaginary parts to get

$$e^{it} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \cos(t) + i\sin(t).$$

Homework: 3, 4, 7, 10, 11, 15, 17, 20, 25, 26, 28, 29, 31, 32, on pp. 81–82.

Euler Equations. An Euler equation is one of the form $x^2y'' + Bxy' + Cy = 0$. The transformation $x = e^t$ transforms the equation into $\ddot{y} + (B-1)\dot{y} + Cy = 0$, where the dot denotes differentiation with respect to t. This equation can be solved using the methods of the last paragraph, upon which substitution of $t = \ln(x)$ yields the solution of the original equation. Consider for example, the equation $x^2y'' + 6xy' + 6y = 0$. Subtituting $x = e^t$, we get $\ddot{y} + 5\dot{y} + 6y = 0$ which has the characteristic equation $\lambda^2 + 5\lambda + 6 = (\lambda + 3)(\lambda + 2) = 0$, with real roots -3, and -2. The general solution is $y = c_1e^{-3t} + c_2e^{-2t}$. Substituting back $x = e^t$, we get $y = c_1x^{-3} + c_2x^{-2}$.

Homework: 1,5, 13, 17, 20, 26, 27, 28, 32, on pp. 85–86.

Undetermined coefficients. This method applies to the non-homogeneous second-order constant coefficients equation $y'' - 2by' + cy = f(x)$, where $f(x)$ is a simple function. We will illustrate with a number of examples:

1. If $f(x) = p_n(x)$ is a polynomial of degree n, substitute $y = x^\alpha q_n(x)$, where q_n is a polynomial of the same degree, and α is such that no term in $x^\alpha q_n(x)$ is a solution of the homogeneous equation. Consider, for example, $y'' - y' - 6y = x - 1$. Substitute $y(x) = ax + b$ and solve for a and b. We get $-a - 6b = -1$, and $-6a = 1$, from which we conclude that $a = -1/6$, $b = 7/36$. Thus, $y(x) = -x/6 + 7/36$ is a solution, and the general solution is $y(x) = c_1e^{-2x} + c_2e^{3x} - x/6 + 7/36$.
2. Similarly, if $f(x) = p_n(x)e^{\beta x}$, we substitute $y = x^\alpha q_n(x)e^{\beta x}$, with α as before. Consider, for example, $y'' - 2y' + 1 = e^x$. Since e^x, and xe^x both are solution of the homogeneous equation,

we substitute $y = ax^2e^x$. We get $a = 1/2$, hence $y(x) = \frac{1}{2}x^2e^x$ is a solution, and the general solution is $y = c_1e^x + c_2xe^x + \frac{1}{2}x^2e^x$.

3. If $f(x) = p_n(x)\cos(\beta x)$, or if $f(x) = p_n(x)\sin(\beta x)$, we substitute
$$y = x^\alpha\left[q_n(x)\cos(\beta x) + r_n(x)\sin(\beta x)\right],$$
with α as before, and r_n another polynomial of degree n.
4. If $f(x) = p_n(x)e^{\beta x}\cos(\gamma x)$, or if $f(x) = p_n(x)e^{\beta x}\sin(\gamma x)$, we substitute
$$y = x^\alpha e^{\beta x}\left[q_n(x)\cos(\gamma x) + r_n(x)\sin(\gamma x)\right],$$
with α, and r_n as before.

There is one more principle: if $f(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$, we may separately look for solutions $y_1(x)$, $y_2(x)$, ... , $y_n(x)$, corresponding to $f_1(x)$, $f_2(x)$, ... , $f_n(x)$, and add these together. Consider, for example, $y'' - y' = 1 + \cos(x)$, which we break down into the two problems $y_1'' - y_1' = 1$, and $y_2'' - y_2' = e^x$. Since $y_1 = a$ is a solution of the first problem, we try $y_1 = ax$. We find that $a = -1$, i.e., $y_1 = -x$ is a solution. For the second problem, we try $y_2 = a\cos(x) + b\sin(x)$, and find that $-\frac{1}{2}\cos(x) - \frac{1}{2}\sin(x)$ is a solution. Thus, a solution of the original problem will be: $y(x) = -x - \frac{1}{2}\cos(x) - \frac{1}{2}\sin(x)$, and the general solution will be $y = c_1x + c_2e^x - x - \frac{1}{2}\cos(x) - \frac{1}{2}\sin(x)$.

Homework: 1, 3, 5, 6, 9, 10, 13, 15, 17, 20, 21, 15, 30, 31, 33, on p. 91.

IX. Application to Oscillations

Models: We will consider three examples:

1. Consider a spring hanging from a ceiling with a mass m attached to its free end. *Hooke's Law* states that if the displacement of the spring is x, then the restorative force on the spring is $-kx$ where k is a constant depending only on the spring. Assuming a linear damping effect, we have a damping force of $-c\dot{x}$ proportional to the velocity, where c is a positive constant. Assuming an external force $f(t)$ acting on the mass, then Newton's second law of motion implies the equation:
$$m\ddot{x} + c\dot{x} + kx = f(t),$$
for the displacement.
2. Consider an RLC electrical circuit, with a variable electromotive force $E(t)$. Kirchoff's voltage law yields: $E(t) = L\,di/dt + Ri + q/C$. Substituting $i = dq/dt$, we get the equation:
$$L\ddot{q} + R\dot{q} + \frac{1}{C}\,q = E(t),$$
for the charge.
3. Consider a string of length l, fixed at its endpoints, undergoing vibrations. Assuming the vibrations are small compared to l, and ignoring damping and other effects, the displacement $u(x,t)$ satisfies the following partial differential equation and boundary conditions:
$$u_{tt} - a^2 u_{xx} = 0; \qquad u(0,t) = u(l,t) = 0.$$
Here a is a constant depending on the tension and constitution of the string. Looking for solutions of the special form $u(x,t) = y(x)\varphi(t)$, we obtain the following:
$$\frac{u_{tt} - a^2 u_{xx}}{a^2 u} = \frac{\ddot{\varphi}}{a^2\varphi} - \frac{y''}{y} = 0.$$
Since $\ddot{\varphi}/\varphi$ is a function of t, and y''/y is a function of x, both must be equal to the same constant μ. Thus, we get the following two ordinary differential equations:
$$y'' - \mu y = 0, \qquad \ddot{\varphi} - a^2\mu\varphi = 0.$$
In addition y must satisfy the boundary conditions $y(0) = y(l) = 0$. This restricts the possible values for μ. If μ is positive, say $\mu = \lambda^2$, then the general solution is $y = c_1\cosh(\lambda x) + c_2\sinh(\lambda x)$ which does not satisfy the boundary conditions. Indeed, $y(0) = 0$ implies that $c_2 = 0$, and then $y(l) = 0$ is impossible. If $\mu = 0$, then the general solution is $y = c_1 + c_2 x$, which again is impossible. If μ is negative, say $\mu = -\lambda^2$, then the general solution is $y = c_1\cos(\lambda x) + c_2\sin(\lambda x)$. This will satisfy the boundary conditions if $c_1 = 0$, and $\lambda l = k\pi$ for some positive integer k. Thus, the only allowable values of λ are $\lambda = k\pi/l$; these are the *modes* of the string. Note that in each mode, the string vibrates according to the equation:
$$\ddot{\varphi} - \omega_k^2\varphi = 0,$$
where $\omega_k = ak\pi/l$. These are the *natural frequencies* of the string. The *base tone* corresponds to the lowest natural frequency $\omega_1 = a\pi/l$; the *overtones* correspond to all the other frequencies. Foe example, if the base tone is pitch C, the first overtone is one octave higher with twice the frequency. The next overtone is an octave and a fifth higher, pitch G, with three times the frequency. This is also the first overtone of the G one octave lower with 3/2 the frequency of the original C. This creates a problem in fixed tuned instruments, such as the piano. Indeed, seven octaves correspond to twelve fifth. Thus, with exact tuning, one would require $2^7 = (3/2)^{12}$, which is not true (about 1.5% off)! The solution: *tempered tuning* where a small compromise is spread throughout the keyboard.

Free Motion: Without the forcing term, the equation is:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0.$$

The roots of the characteristic equation are:

$$\lambda = \frac{1}{2m}(-c \pm \sqrt{c^2 - 4km}).$$

There are three cases:

1. *Overdamping:* $c^2 > 4km$. The characteristic equation has two distinct real roots:
$$\lambda_1 = \frac{1}{2m}(-c + \sqrt{c^2 - 4km}), \qquad \lambda_2 = \frac{1}{2m}(-c - \sqrt{c^2 - 4km}).$$
Both are negative, since $\sqrt{c^2 - 4km} < c$, and it is possible to show that, regardless of the initial conditions, the solution:
$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$
has at most one oscillation, i.e., $\dot{x}$ has at most one zero.
2. *Critical damping:* $c^2 = 4km$. In this case, the general solution is:
$$x = (c_1 + c_2 t)e^{-ct/2m}.$$
and again, the solution has at most one oscillation.
3. *Underdamping:* $c^2 < 4km$. The general solution is now:
$$x = e^{-ct/2m}\big[c_1 \cos(\omega t) + c_2 \sin(\omega t)\big],$$
where $\omega = \sqrt{4km - c^2}/2m$. Now there are infinitely many oscillations, i.e., $\dot{x}$ has infinitely many zeros.

Note that in all three cases $\dot{x} \to 0$ as $t \to \infty$.

Forced Motion: We will only consider a periodic external force: $f(t) = A\cos(\omega t)$, where A is the amplitude, and ω the frequency of the force. The equation is:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{A}{m}\cos(\omega t).$$

Using undetermined coefficients, we substitute $x = a\cos(\omega t) + b\sin(\omega t)$ in the equation to get:

$$(-ma\omega^2 + b\omega c + ak - A)\cos(\omega t) = (mb\omega^2 + a\omega c - bk)\sin(\omega t).$$

Since this must hold for all t, and since the functions $\sin(\omega t)$ and $\cos(\omega t)$ are linearly independent, we get:

$$(k - m\omega^2)a + \omega c b = A, \qquad -\omega c a + (k - m\omega^2)b = 0.$$

Solving for a and b, we get:

$$a = \frac{mA(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \qquad b = \frac{A\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2},$$

where $\omega_0 = \sqrt{k/m}$. As before, we have overdamped, critically damped, or underdamped forced motion. We will only consider two specific examples:

1. *Overdamped motion:* Consider the initial value problem: $\ddot{x} + 6\dot{x} + 5x = \cos(t)$, $x(0) = 0$, $\dot{x}(0) = 0$. The solution is:
$$x = \frac{5}{104}e^{-5t} - \frac{1}{8}e^{-t} + \frac{1}{13}\cos(t) + \frac{3}{26}\sin(t).$$

Note that the only oscillatory motion, and also the only motion remaining in the long run, is the one driven by force.

2. *Underdamped motion:* Consider the initial value problem: $\ddot{x} + 2\dot{x} + 2x = \cos(2t)$, $x(0) = 0$, $\dot{x}(0) = 0$. The solution is:
$$x = e^{-t}\left(\frac{1}{10}\cos(t) - \frac{3}{10}\sin(t)\right) - \frac{1}{10}\cos(2t) + \frac{1}{5}\sin(2t).$$
Again, in the long run the only motion is the one driven by the force. However, for a while, the natural oscillations, with period 1, are excited by the driving force.

Resonance: Consider an undamped oscillator with the natural frequency equal to the input frequency:
$$\ddot{x} + \omega_0^2 x = A\cos(\omega_0 t).$$
The general solution is:
$$x = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{A}{2\omega_0} t\sin(\omega_0 t).$$
With the initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, the solution is:
$$x = \frac{A}{2\omega_0} t\sin(\omega_0 t).$$
Note that the amplitude of the oscillations increase without bound.

Amplitude Modulation: Consider an undamped oscillator with the natural frequency very close to the input frequency:
$$\ddot{x} + \omega_0^2 x = A\cos(\omega t),$$
where $\omega - \omega_0$ is very small (compared to ω_0). The general solution is:
$$x = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{A}{\omega_0^2 - \omega^2}\cos(\omega t).$$
With the initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, the solution is:
$$x = \frac{A}{\omega_0^2 - \omega^2}\big(\cos(\omega t) - \cos(\omega_0 t)\big).$$
This can be rewritten as:
$$x = \frac{2A}{\omega_0^2 - \omega^2}\sin\big((\omega_0 + \omega)t/2\big)\sin\big((\omega_0 - \omega)t/2\big).$$
This can be viewed as an oscillation with a frequency $(\omega_0 + \omega)/2)$, the average of ω_0 and ω, and varying, or modulated, amplitude. The amplitude oscillates with frequency $(\omega_0 - \omega)/2$, a much lower frequency. This is used by piano tuners, to fine tune strings slightly off exact tuning. Two pitches very close together will exhibit a so-called *beat* whose frequency the tuner is able to count.

Homework: 1, 2, 3, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 22, 25, 26, 27, 28, 30, 31, 32, 33, on pp. 105–107; and 1, 2, 3, on p. 109.

X. Variation of Parameters

We now return to the general non-homogeneous problem: $y'' + p(x)y' + q(x)y = f(x)$. We assume that a fundamental system of solutions y_1, y_2 of the homogeneous equation has already been found. The method of variation of parameters, will enable us to find a particular solution of the non-homogeneous equation. We look for a solution of the form $y = uy_1 + vy_2$, where u and v are functions satisfying $u'y_1 + v'y_2 = 0$. Differentiating twice, we obtain $y' = uy_1' + vy_2'$, and $y'' = u'y_1' + v'y_2' + uy_1'' + vy_2''$, which after substitution into the equation yields together with the condition above, the system:

$$y_1u' + y_2v' = 0$$
$$y_1'u' + y_2'v' = f.$$

Since y_1, and y_2 are linearly independent, their Wronskian is non-zero, hence this system can be solved for u' and v'.

Example: Find the general solution of $y'' - 3y' + 2y = \cos(e^{-x})$. The characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$, with roots $\lambda = 1$, and $\lambda = 2$. Thus, two linearly independent solutions are $y_1 = e^x$, and $y_2 = e^{2x}$. Following the method above, we get:

$$e^x u' + e^{2x} v' = 0$$
$$e^x u' + 2e^{2x} v' = \cos\left(e^{-x}\right).$$

Solving these equations, we get: $u' = -e^{-x}\cos(e^{-x})$, $v' = e^{-2x}\cos(e^{-x})$. Integrating these, we get: $u = \sin(e^{-x})$, $v = -e^{-x}\sin(e^{-x}) - \cos(e^{-x})$. Thus, a particular solution of the non-homogeneous equation is:

$$y = ue^x + ve^{2x} = -e^{2x}\cos\left(e^{-x}\right),$$

and the general solution is:

$$y = c_1e^x + c_2e^{2x} - e^{2x}\cos\left(e^{-x}\right).$$

Homework: 1, 2, 4, 7, 8, 9, 14, 20, 22, 26, 27, 28, 29, on pp. 94–95.

XI. Numerical Methods

All the methods will be illustrated on the first order initial value problem:

$$y' = f(x,y), \qquad y(0) = y_0.$$

Euler's Method: The idea is to approximate the values of f at $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ..., $x_n = x_0 + nh$ by using its first degree Taylor polynomial. Thus, $y_1 = y_0 + y'(x_0)(x_1 - x_0) = y_0 + hf(x_0, y_0)$, $y_2 = y_1 + hf(x_1, y_1)$, ..., $y_{n+1} = y_n + hf(x_n, y_n)$. Consider, for example the initial value problem:

$$y' = y\sin(x); \qquad y(0) = 1.$$

The solution is:

$$y = e^{1-\cos(x)}.$$

We use Euler's method to approximate the solution with three different step sizes h, and number of iterations N: (i) $h = 1/5$, $N = 20$; (ii) $h = 1/10$, $N = 40$; (iii) $h = 1/20$, $N = 80$. Defining the *global discretization error* by:

$$E(h) = \max_{1\leqslant k\leqslant N} |y_k - y(x_k)|,$$

we obtain in the cases above: (i) $E(1/5) \leqslant 1.75$; (ii) $E(1/10) \leqslant 0.93$; (iii) $E(1/20) \leqslant 0.48$.

Error Analysis for Euler's Method: We now show that if f and f_y are continuous, and if the solution y has a continuous second order derivative y'', then $|E(h)| = O(h)$, i.e., $E(h) \leqslant Ch$ for some constant C independent of h. Here, it is understood that the end points $a = x_0$, and $b = x_N$ are fixed, and thus in particular, the size $B = b - a$ of the interval is also fixed. We define the *local discretization error at x* by:

$$L(x,h) = \frac{1}{h}\big[y(x+h) - y(x)\big] - f(x, y(x)).$$

Note that if $y_k = y(x_k)$, then

$$y(x_{k+1}) - y_{k+1} = hL(x_k, h).$$

Using a second order Taylor expansion, we obtain:

$$L(x,h) = \frac{1}{2}hy''(\xi),$$

for some $x_0 < \xi < x_0 + Nh$. Since y'' is continuous on the closed interval $[x_0, x_0 + Nh]$, we may assume that $|y''(\xi)| \leqslant M$ for some constant M, i.e., we conclude that

$$|L(x,h)| \leqslant \frac{1}{2}hM.$$

Define the *local discretization error* by:

$$L(h) = \max_{x_0\leqslant x\leqslant x_0+Nh} |L(x,h)|.$$

We have just derived the bound:

$$L(h) \leqslant \frac{1}{2}hM.$$

Note that $y(x_k + h) = hL(x_k + h) + hf(x_k, y(x_k)) + y(x_k)$, hence

$$\begin{aligned} y(x_{k+1}) - y_{k+1} &= y(x_k + h) - y_k - hf(x_k, y_k) \\ &= y(x_k) - y_k + hL(x_k, h) + h\big[f(x_k, y(x_k)) - f(x_k, y_k)\big]. \end{aligned}$$

Since f and f_y are continuous, we can use the mean value theorem to obtain that

$$|f(x, y(x_k)) - f(x_k, y_k)| = |f_y(x_k, \eta)(y(x_k) - y_k)| \leqslant R\,|y(x_k) - y_k|\,,$$

where η is between $y(x_k)$ and y_k, and $R \geqslant |f_y|$. Let $e_k = y(x_k) - y_k$, then we have obtained:

$$|e_{k+1}| \leqslant (1 + hR)\,|e_k| + hL(h) \leqslant C\,|e_k| + K.$$

where $C = 1 + hR$, and $K = Mh^2/2$. By iterating this inequality, we deduce:

$$|e_{k+1}| \leqslant (1 + C + \cdots + C^{k-1})K \leqslant (1 + C + \cdots + C^{N-1})K.$$

Note that $N = B/h$, hence:

$$C^N = (1 + hR)^{B/h} \to e^{RB},$$

and thus, there is a number P, independent of h, such that $C^k \leqslant P$ for $1 \leqslant k \leqslant N - 1$. We conclude:

$$|e_{k+1}| \leqslant \frac{1}{2}NPMh^2 = \frac{1}{2}BPh.$$

Since this is true for all $0 \leqslant k \leqslant N - 1$, we finally have obtained:

$$E(h) \leqslant \frac{1}{2}BPMh.$$

Homework: 2, 6, 7, 8, 9, 11, 13, 14, on pp. 202–203.

Other One-Step Methods: *The Second Order Taylor Method* is similar to the Euler Method, but uses the second, instead of the first, degree polynomial: $y(x) \approx y(x_0) + y'(x_0)(x - x_0) + \frac{1}{2}y''(x_0)(x - x_0)^2$. The second derivative is computed using the equation $y' = f(x, y)$: $y'' = f_x + f_y y' = f_x + f_y f$. Thus, we obtain:

$$y_{k+1} = y_k + hf(x_k, y_k) + \frac{1}{2}h^2\big[f_x(x_k, y_k) + f_y(x_k, y_k)f(x_k, y_k)\big].$$

Rewrite the previous formula as:

$$y_{k+1} = y_k + h\left[f(x_k, y_k) + \frac{h}{2}\big(f_x(x_k, y_k) + f(x_k, y_k)f_y(x_k, y_k)\big)\right].$$

Note that $f(x_k, y_k) + \frac{h}{2}\big(f_x(x_k, y_k) + f(x_k, y_k)f_y(x_k, y_k)\big)$ is the first degree Taylor polynomial for $f(x_k + h/2, y_k + hf_k/2)$ where $f_k = f(x_k, y_k)$. The *Modified Euler Method* uses this idea:

$$y_{k+1} = y_k + hf\left(x_k + \frac{h}{2}, y_k + \frac{hf(x_k, y_k)}{2}\right).$$

The *RK4 Method* elaborates on that idea:

$$y_{k+1} = y_k + \frac{16}{h}[W_{k1} + 2W_{k2} + 2W_{k3} + W_{k4}],$$

where

$$\begin{aligned}
W_{k1} &= f(x_k, y_k),\\
W_{k2} &= f\left(x_k + \frac{h}{2}, y_k + \frac{hW_{k1}}{2}\right),\\
W_{k3} &= f\left(x_k + \frac{h}{2}, y_k + \frac{hW_{k2}}{2}\right),\\
W_{k4} &= f\left(x_k + h, y_k + hW_{k3}\right).
\end{aligned}$$

All three methods are *one-step method*, in the sense that y_{k+1} is computed using y_k, and all are second order, in the sense that $E(h) = O(h^2)$.